Non-backtracking Eigenvalues: X-Centrality and Node Immunization

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The *non-backtracking matrix* (or *NB-matrix* for short) has many applications in network science, in particular in node centrality and epidemic thresholds. Let λ be its leading eigenvalue. In epidemiology, $1/\lambda$ is a good approximation for the epidemic threshold of certain network dynamics. In this work, we introduce efficient ways of identifying which nodes have the largest impact on λ. We do so by studying the spectrum of the NB-matrix after a node is removed from the graph. From this analysis we derive two new centrality measures: *X-degree* and *X-non-backtracking centrality*.

Given a graph G, the *NB-matrix* B is indexed in the rows and columns by directed edges, and it is defined by $B_{k\to l,i\to j} = \delta_{jk}\left(1-\delta_{il}\right)$, where δ is the Kronecker delta. Consider a node c, and let λ_c be the leading eigenvalue of the NB-matrix of the graph after c has been removed. We call $\lambda - \lambda_c$ the *eigen-drop* induced by c. Computing the eigen-drop is computationally expensive. Our spectral analysis naturall yields two new proxy measures that are highly correlated with the eigen-drop (see Fig. 1) and faster to compute.

First, the *X-degree* centrality of node c is defined as $\left(\sum_i a_{ci}\left(k_i-1\right)\right)^2 - \sum_i a_{ci}\left(k_i-1\right)^2$, where (a_{ij}) is the adjacency matrix of G and k_i is the degree of node i. Second, the *X-non-backtracking centrality* of c is defined as $\left(\sum_i a_{ci}\mathbf{v}_i\right)^2 - \sum_i a_{ci}\left(\mathbf{v}_i\right)^2$, where $\mathbf{v}_i$ is the so-called non-backtracking centrality of i.[5] Note the similarity of these expressions: they are both defined as a function of the second moment of the distribution of a node's neighbors' (degree or non-backtracking) centralities. Note also the similarity between *X-degree* and the Collective Influence measure.[6] We think of *X-degree* as a second-order aggregation of the excess degree values $(k_i - 1)$, whereas CI is a first-order aggregation. Importantly, our two measures are derived in ways entirely different from the derivation of CI, based on a novel spectral perturbation analysis. These similarities will be explored in future lines of research.

We perform extensive experimentation with targeted immunization strategies derived from these centrality measures, whose objective is to reduce λ as much as possible. Our algorithms have average time complexity that is linear in the number of nodes in G. Additionally, we further explore the implications of the X-centrality framework, whose formulae indicate that nodes whose neighbors' centralities have small variance (i.e. large second moment) will have a large influence on λ.

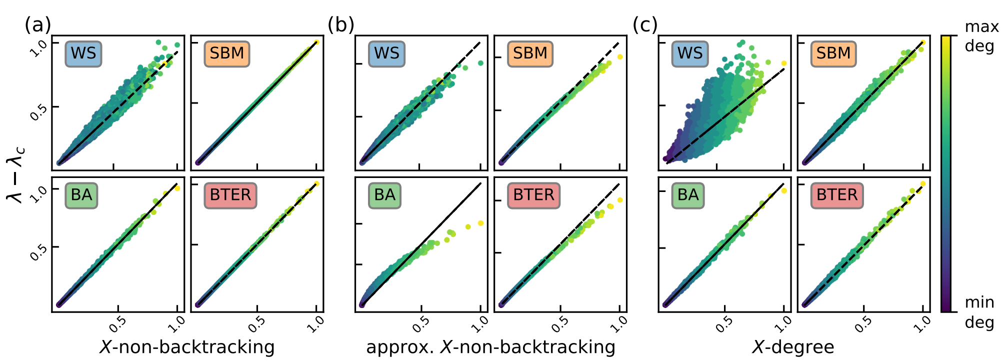

Fig. 1. Eigen-drop and X-Centrality are highly correlated. Each marker represents one node randomly sampled from a graph. WS: Watts-Strogatz, SBM: Stochastic-Block Model, BA: Barabási-Albert, BTER: Block Two–Level Erdős-Rényi. We sampled 100 graphs per model, each with 10^5 nodes. Dashed black lines are linear regression lines.

[5] T. Martin, X. Zhang, and M.E.J. Newman. *Localization and centrality in networks.* Phys. Rev. E, 90(052808), 2014.
[6] F. Morone, H. A. Makse, *Influence maximization in complex networks through optimal percolation.* Nature, 524(7563), 65-68. 2015.